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The reflection of light by planar stratified media: the groupoid of amplitudes and a phase 'Thomas precession'

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Abstract. The reflection coefficient of light on stratified planar structures can be obtained by postulating the use of a complex generalization of Einstein's addition theorem for parallel velocities.

The algebraic properties of the 'composition law of amplitudes' show that the set of all complex amplitudes of the electromagnetic field in heterostructures forms a *weakly associative-commutative groupoid.* The first concrete application of this abstract concept was found only in 1988 in special relativity. This work exhibits another example in a quite different field of physics. It also puts into evidence that the 'phase rotation' of light in stratified planar structures is to be considered as a 'Thomas rotation'.

1. Introduction

The overall reflection coefficient of any number of isotropic media can be directly obtained, whatever may be the number of interfaces, by postulating the generalization in the complex plane of Einstein's well known relativistic composition law of parallel velocities [1]. Noting R_1 and R_2 the complex (including phases) reflection coefficients of two interfaces, this composition law, which we denote $\oplus$, is defined by (the meaning of the overbar on R will be defined below)

$$\mathscr{R}_2 = R_1 \oplus R_2 = \frac{R_1 + R_2}{1 + \bar{R}_1 R_2} \tag{1}$$

similar to the relativistic composition law

$$\mathscr{V}_2 = V_1 \oplus V_2 = \frac{V_1 + V_2}{1 + V_1 V_2 / c^2}. \tag{2}$$

The only difference between (1) and (2) lies in the fact that the R_i in (1) are complex quantities whereas the velocities V_i in (2) are real quantities.

As shown in [1], equation (1) provides a useful mathematical tool in the calculation of heterostructures. It also gives new insight into special relativity. The composition law of velocities does no more appear as a peculiar property of special relativity but as the expression, in the particular case of dynamics, of a more general 'composition law' of physics.

The physical meaning of the 'composition law of amplitudes' is quite similar to that of the 'composition law of velocities' in special relativity:

> As is well known, equation (2) shows that no matter what values we give to V_1 and V_2, subject only to $|V_1|<c$ and $|V_2|<c$, then the value of $|\mathcal{V}_2|$ cannot exceed the speed of light c. In the same way, no matter what values the reflection amplitudes R_1 and R_2 (subject only to $|R_1|<1$ and $|R_2|<1$) have, the overall reflection coefficient $\mathcal{R}_2$ cannot exceed unity (the incident amplitude).

Iterating the composition law of amplitudes (1) directly leads to the overall reflection coefficient of a system made with any number of interfaces [1]. However, as underlined in that previous work, the composition law of amplitudes, in the complex plane, is:

(*a*) neither associative (for example, the expression for the overall reflection coefficient is $\mathcal{R}_n=(R_1\oplus(R_2\oplus(R_3\oplus(\ldots\oplus R_n))))$ and by no means $((((R_1\oplus R_2)\oplus R_3)\oplus\ldots)\oplus R_n)$;

(*b*) nor commutative (in general $(R_1\oplus R_2)$ and $(R_2\oplus R_1)$ have different phases although their magnitude are the same).

In order to simplify calculations, which could be rather difficult because of this, our aim is to study the underlying algebraic formalism of the composition law of amplitudes (1). We show that the set of all the possible complex amplitudes of light in stratified planar structures form a *weakly associative-commutative groupoid* with the operation $\oplus$. These grouplike properties of amplitudes are quite similar to that of the relativistic admissible velocities which have been studied by Ungar [2-6], who suggested that such a groupoid be called a gyrogroup. We refer to his notation and proceed as he did.

We show that the non-commutativity and the non-associativity of the composition law of amplitudes result from the presence, in the expression of R_t, of a phase term which exactly plays the part of the Thomas precession in special relativity. Because of this, we consider the 'phase rotation' which appears in calculations of multilayers as a 'Thomas phase rotation' (or precession). Let us note that the Thomas rotation, which is generally studied as an isolated result in special relativity, appears here as a more general notion than we are used to.

It is to be noted that, although discovered in 1965 by Karzel [7], who named it a K loop, the first concrete example of weakly associative-commutative groupoid was only discovered in 1988 by Ungar [8] in his study on the parametrization of the Lorentz transformation group. Our present result provides another concrete example of a gyrogroup in a quite different field of physics.

2. The use of Einstein's addition law of parallel velocities in studies of reflection by stratified planar structures

Let us consider a planar stratified medium made of n planar parallel interfaces. Using (1) and noting $R_j\equiv R_{j,j+1}$, the complex (including phases) reflection coefficient of the interface between the two media n_j and n_{j+1}, the overall reflection coefficient of the total structure can be written [1] as

$$\mathcal{R}_{(1,2,\ldots,n)}\equiv\mathcal{R}_n=R_1\oplus(R_2\oplus(\ldots\oplus R_n)). \tag{3}$$

(For clarity the complete notation $\mathscr{R}_{(i,j,\ldots,n)}$, specifying inside the subscript the serial number of each interface, will be used only when necessary.)

In (1) and (3) $R_j \equiv R_{j,j+1}$ is defined by

$$R_j \equiv R_{j,j+1} = r_{j,j+1} \exp[-2i(\beta_1 + \beta_2 + \ldots + \beta_j)] \tag{4}$$

$$\bar{R} \equiv \bar{R}_{j,j+1} = r_{j,j+1} \exp[+2i(\beta_1 + \beta_2 + \ldots + \beta_j)]. \tag{5}$$

Let us emphasize that the bar on R denotes the change of β_j into $-\beta_j$ in (4) and (5). This operation corresponds to taking the complex conjugate of R_j only in the case when $r_{j,j+1}$ is real.

R_j corresponds (figure 1) to the reflection coefficient of the wave on the $(j, j+1)$ interface; $r_{j,j+1}$ is the Fresnel reflection coefficient of that interface (for simplicity, the appropriate subscripts p and s corresponding to the polarization of light have been dropped from all equations); the phaseshift β_j corresponds to the effect of propagation of the field through the same homogeneous layer of index of refraction n_j between the two interfaces located at z_{j-1} and $z_j = z_{j-1} + d_j$

$$\beta_j = \frac{2\pi}{\lambda} n_j \cos\theta_j \qquad (z_j - z_{j-1}) = q_j d_j. \tag{6}$$

θ_j is the angle between the direction of propagation of the wave in the layer n_j and the perpendicular to its boundaries (the z-axis). The meaning of q_j is obvious from (6) and corresponds to the z-component of the wavevector normal to the interface, in the corresponding medium.

Let us note, that, using equation (1), equation (3) can also be written

$$\mathscr{R}_{(1,2,\ldots,n)} = R_1 \oplus \mathscr{R}_{(2,3,\ldots,n)} = \frac{R_1 + \mathscr{R}_{(2,3,\ldots,n)}}{1 + \bar{R}_1 \mathscr{R}_{(2,3,\ldots,n)}}. \tag{7}$$

We emphasize that, in the above equation, R_1 corresponds to the reflection coefficient of the wave on the first interface and that $\mathscr{R}_{(2,3,\ldots,n)}$ corresponds to the overall reflection coefficient of a planar stratified structure made with interfaces, $2, 3, \ldots, n$.

For clarity, let us illustrate the use of (3) (or (7)) by giving some examples. In the case of two interfaces, equation (1) directly leads to

$$\mathscr{R}_{(1,2)} \equiv \mathscr{R}_2 = \frac{R_1 + R_2}{1 + \bar{R}_1 R_2} \tag{8}$$

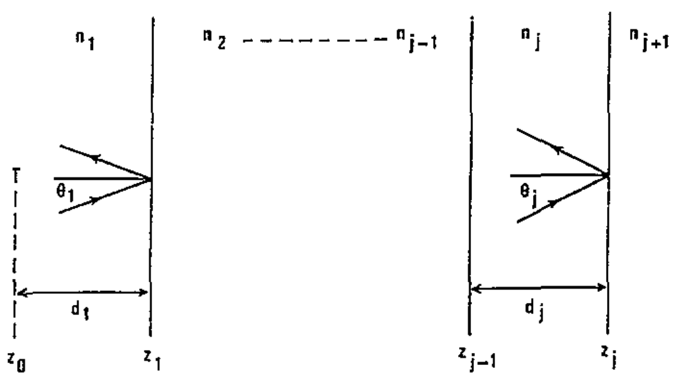

Figure 1. Illustration of notation in the case of n parallel planar surfaces.

which, using (4) and (5) gives the well known result

$$\mathscr{R}_{(1,2)} \equiv \mathscr{R}_2 = \frac{r_{12} + r_{23}\,\mathrm{e}^{-\mathrm{i}2\beta_2}}{1 + r_{12} r_{23}\,\mathrm{e}^{-\mathrm{i}2\beta_2}}\,\mathrm{e}^{-\mathrm{i}2\beta_1}. \tag{9}$$

In the case of three interfaces $\mathscr{R}_3$ can be written in the same way

$$\mathscr{R}_{(1,2,3)} \equiv \mathscr{R}_3 = \frac{R_1 + R_2 + R_3 + R_1 \bar{R}_2 R_3}{1 + \bar{R}_1 R_2 + \bar{R}_1 R_3 + \bar{R}_2 R_3} \tag{10}$$

which, together with (4) and (5) leads to the usual expression [9]

$$\mathscr{R}_3 = \frac{(r_{12} + r_{23}\,\mathrm{e}^{-2\mathrm{i}\beta_2}) + (r_{12} r_{23} + \mathrm{e}^{-2\mathrm{i}\beta_2}) r_{34}\,\mathrm{e}^{-2\mathrm{i}\beta_3}}{(1 + r_{12} r_{23}\,\mathrm{e}^{-2\mathrm{i}\beta_2}) + (r_{23} + r_{12}\,\mathrm{e}^{-2\mathrm{i}\beta_2}) r_{34}\,\mathrm{e}^{-2\mathrm{i}\beta_3}}\,\mathrm{e}^{-2\mathrm{i}\beta_1}. \tag{11}$$

In the same way

$$\mathscr{R}_4 = \frac{R_1 + R_2 + R_3 + R_4 + R_1 \bar{R}_2 R_3 + R_1 \bar{R}_2 R_4 + R_1 \bar{R}_3 R_4 + R_2 \bar{R}_3 R_4}{1 + \bar{R}_1 R_2 + \bar{R}_1 R_3 + \bar{R}_1 R_4 + \bar{R}_2 R_3 + \bar{R}_2 R_4 + \bar{R}_3 R_4 + \bar{R}_1 R_2 \bar{R}_3 R_4}. \tag{12}$$

Note that although never considered (as far as I know) in the literature, equations (3), (10) and (12) are also valid in special relativity. Einstein's additional theorem for n, 3 or 4 parallel velocities V_j can be shown [1] to be given by equations (3), (10) and (12) respectively after having changed R_i into V_i.

Note that the expression (3) of $\mathscr{R}_n$ for any value of n can be expressed in a polynomial form [10] by introducing a complex generalization of the so-called 'elementary symmetric functions' of the mathematical theory of polynomials [11, 12].

3. Algebraic properties of the composition law of amplitudes

In special relativity, the observer K_0 can only measure the velocity V_1 of K_1 (relative to him, K_0) and that $\mathscr{V}_2$ of K_2 (relative to him, K_0). Since he is not in the inertial frame K_1, he cannot *measure* the velocity V_2 of K_2 (relative to K_1). The only way for observer K_0 to express the velocity V_2 of K_2, *relative to observer* K_1 from that observer's own knowledge of V_1 and $\mathscr{V}_2$, is not by the usual addition law, but by the composition law (2).

In optics let us consider three successive media. An observer K_0 (*'immersed' in a medium defined by its refractive index* n_1) can measure the reflection coefficient $R_1 \equiv R_{1,2}$ (relative to his own medium n_1) of light on the interface (1, 2). He can also measure the overall reflection coefficient $\mathscr{R}_2 \equiv \mathscr{R}_{(1,2)}$ (also relative to his own medium n_1) *of the total structure* n_1-n_2-n_3. Not being in the medium n_2 himself, he cannot measure the reflection coefficient $R_2 \equiv R_{2,3}$ of light propagating in the medium n_2 and falling on the interface n_2-n_3. The only way for him to deduce $R_2 = R_{2,3}$ (*relative to an observer* K_1 *inside the medium* n_2) from that observer's knowledge of R_1 and $\mathscr{R}_2$, is not by the usual addition, but by the composition law (1). We thus have

$$\mathscr{R}_2 = \frac{R_1 + R_2}{1 + \bar{R}_1 R_2} \;\Rightarrow\; R_2 = \frac{(-R_1) + \mathscr{R}_2}{1 + \overline{(-R_1)}\mathscr{R}_2} \tag{13}$$

which can be written in the more compact form

$$\mathscr{R}_2 = R_1 \circ R_2 \;\Rightarrow\; R_2 = (-R_1) \circ \mathscr{R}_2. \tag{14}$$

The knowledge of R_2 can thus be obtained from that of $\mathscr{R}_2$ and R_1.

In the case of three interfaces we have in the same way, using (3)

$$\mathscr{R}_3 = R_1 \oplus (R_2 \oplus R_3) = R_1 \oplus \mathscr{R}_{(2,3)} \Rightarrow \mathscr{R}_{(2,3)} = (R_2 \oplus R_3) = (-R_1) \circ \mathscr{R}_3 \tag{15}$$

so that, we can easily deduce the knowledge of $\mathscr{R}_{(2,3)}$ from that of R_1 and $\mathscr{R}_3 \equiv R_{(1,2,3)}$. However, in this example, the problem of calculating $\mathscr{R}_{(1,2)}$ from a knowledge of $R_3 \equiv R_{3,4}$ and of $\mathscr{R}_3 \equiv \mathscr{R}_{(1,2,3)}$ would not be straightforward. It would suppose, in fact, that we express $\mathscr{R}_3$ with respect to $(R_1 \circ R_2)$ and R_3 instead of expressing it with respect to R_1 and $(R_2 \oplus R_3)$ as in (15). This is not obvious because of the non-associativity of the composition law (1), which can easily be seen from the two results

$$R_1 \circ (R_2 \circ R_3) \equiv R_1 \oplus \mathscr{R}_{(2,3)} = \frac{R_1 + R_2 + R_3 + R_1 \bar{R}_2 R_3}{1 + \bar{R}_1 R_2 + \bar{R}_1 R_3 + \bar{R}_2 R_3} \tag{16}$$

$$(R_1 \circ R_2) \circ R_3 \equiv \mathscr{R}_{(1,2)} \oplus R_3 = \frac{R_1 + R_2 + R_3 + \bar{R}_1 R_2 R_3}{1 + R_1 \bar{R}_1 + \bar{R}_1 R_3 + \bar{R}_2 R_3}. \tag{17}$$

In order to show how to solve equation (15) for $\mathscr{R}_{(1,2)}$, our aim is to study algebraic properties of the composition law (1). This will lead us to define the gyrogroup of complex amplitudes in stratified planar structures. Let us note that this part explicitly refers to Ungar's paper [6].

(*a*) The composition law of amplitudes is a weak-commutative law. Let us write (1) in the form

$$R_1 \oplus R_2 \equiv \mathscr{R}_{(1,2)} = \frac{R_1 + R_2}{1 + \bar{R}_1 R_2} \equiv \rho \exp \mathrm{j}\varphi_{(R_1 \oplus R_2)} \tag{18}$$

$$R_2 \oplus R_1 = \mathscr{R}_{(2,1)} = \frac{R_1 + R_2}{1 + R_1 \bar{R}_2} \equiv \rho \exp \mathrm{j}\varphi_{(R_2 \oplus R_1)} \tag{19}$$

so that, noting

$$[R_1; R_2] = \exp \mathrm{j}(\varphi_{R_1 \oplus R_2} - \varphi_{R_2 \oplus R_1}) = [R_2; R_1]^{-1} \tag{20}$$

we have

$$(R_1 \oplus R_2) = (R_2 \oplus R_1)[R_1; R_2] \tag{21}$$

this result expresses the weak commutative property of the composition law of amplitudes.

Although distinct, the composite amplitudes $(R_1 \oplus R_2)$ and $(R_2 \oplus R_1)$ have the same magnitude, so that there exist a 'phase rotation' $[R_1 R_2]$ taking $(R_2 \oplus R_1)$ onto $(R_1 \oplus R_2)$. Using (4) and (5) in (18) and (19), this phase rotation comes out as

$$\tan(\varphi_{R_1 \oplus R_2} - \varphi_{R_2 \oplus R_1}) = \frac{2(1 + r_1 r_2 \cos 2\beta_2) r_1 r_2 \sin 2\beta_2}{(1 + r_1 r_2 \cos 2\beta_2)^2 - r_1^2 r_2^2 \sin^2 2\beta_2}. \tag{22}$$

The expression (22) is quite similar to the one of 'Thomas precession' as given by [6, equation (37*a*)]. In fact, changing its notation k into $k = 1/K$, the 'Thomas precession' ε is defined by

$$\tan \varepsilon = \frac{-2(1 + K \cos \theta) K \sin \theta}{(1 + K \cos \theta)^2 - K^2 \sin^2 \theta} \tag{23}$$

which is to be compared with (22). Let us, however, underline the following point. Einstein's composition law for parallel velocities (2) is commutative; the Thomas precession appears only when the velocities are not parallel. In the case of multilayers, the 'phase precession' already appears in (1) because of the complex character of the reflection coefficients.

(*b*) The phase rotation $[R_1; R_2]$ respects the binary operation '$\oplus$'.

$$(R_1\oplus R_2)[R_1; R_2] = (R_1[R_1; R_2])\oplus(R_2[R_1; R_2]). \tag{24}$$

This can be shown by using (1) and by noting that in the denominator of equation (1) we can write

$$R_1R_2 = \bar{R}_1[R_1; R_2]^{-1}[R_1; R_2]R_2 = \overline{R_1[R_1; R_2]}[R_1; R_2]R_2. \tag{25}$$

(*c*) The composition law of amplitudes is a right and left weak-associative law defined by

$$R_1\oplus(R_2\oplus R_3) = (R_1\oplus R_2)\oplus(R_3[R_1; R_2]) \tag{26}$$

$$(R_1\oplus R_2)\oplus R_3 = R_1\oplus(R_2\oplus R_3[R_2; R_1]). \tag{27}$$

To demonstrate (26) note that (16) can be written

$$\begin{aligned} R_1\oplus(R_2\oplus R_3) &= \frac{(R_2+R_1)/(1+\bar{R}_2R_1)+R_3}{1+R_3(\bar{R}_1+\bar{R}_2)/(1+\bar{R}_1R_2)}\,\frac{1+R_1\bar{R}_2}{1+\bar{R}_1R_2} \\ &= \frac{(R_2\oplus R_1)+R_3}{1+\overline{R_2\oplus R_1}R_3}\,\frac{R_1\oplus R_2}{R_2\oplus R_1} \end{aligned} \tag{28}$$

so that we successively get from (28)

$$\begin{aligned} R_1\oplus(R_2\oplus R_3) &= ((R_2\oplus R_1)\oplus R_3)[R_1; R_2] \\ &= (R_2\oplus R_1)[R_1; R_2]\oplus R_3[R_1; R_2] \\ &= (R_1\oplus R_2)\oplus R_3[R_1; R_2] \end{aligned} \tag{29}$$

which establishes the validity of (26). (We have used equations (1) and (21) in the first equality of (29), equation (24) in the second one and again equation (21) in the last equality.)

In the same way, by changing R_3 into $R_3[R_1; R_2]^{-1}$ in (26) and by noting that $[R_1; R_2]^{-1} = [R_2; R_1]$ (see (20)) we obtain (27).

(*d*) Using the same notation as in [6], we can sum up the group-like properties underlying the set of all complex amplitudes in stratified planar structures. Denoting A the set of complex amplitudes of the electromagnetic field in stratified planar structures, we have for all $R_1, R_2, R_3 \in A$ (W is used for the weakly associative-commutative groupoid):

W1	$R_1\oplus R_2\in A$	closure property
W2*a*	$R_1\oplus(R_2\oplus R_3) = (R_1\oplus R_2)\oplus(R_3[R_1; R_2])$	right weak associative law
W2*b*	$(R_1\oplus R_2)\oplus R_3 = R_1\oplus(R_2\oplus R_3[R_2; R_1])$	left weak associative law
W3	$(R_1\oplus R_2) = (R_2\oplus R_1)[R_1; R_2]$	weak commutative law
W4	$0\oplus R_1 = R_1\oplus 0 = R_1$	existence of an identity element
W5	$(-R_1)\oplus R_1 = R_1\oplus(-R_1) = 0$	existence of inverse.

Note that axiom W1 asserts that the group-like operation $\oplus$ is closed in A, that is, the composition $R_i\oplus R_j$ of any two elements of A itself belongs to A. W4 and W5 are straightforward from (1).

Equations (W1)–(W5) exhibit the basic properties of a weakly associative–commutative groupoid with the group operation given by the composition of amplitudes (1). Reference [6] suggests such a structure be called a gyrogroup.

To these five relations let us add (or recall) some useful properties of the 'Thomas phase rotation'

W6 $[R_1; R_2]^{-1} = [R_2; R_1]$

W7 $[R_i; R_j] = 1$ when $R_i \oplus R_j = R_j \oplus R_i$

W8 $[R_1[R_i; R_j]; R_2[R_i, R_j]] = [R_1; R_2]$

W9 $(R_1 \oplus R_2)[R_i, R_j] = R_1[R_i; R_j] \oplus R_2[R_i; R_j]$

W10 $[R_1 \oplus R_2; R_2] = [R_1; R_2]$ loop property.

W6 and W9 correspond to (20) and (24) respectively. W7 follows from (20).

In order to demonstrate W8 we can write, using (20)

$$
\begin{aligned}
&[R_1[R_iR_j]; R_2[R_iR_j]] \\
&\quad = \exp \mathrm{j}(\varphi_{R_1[R_iR_j]\oplus R_2[R_iR_j]} - \varphi_{R_2[R_iR_j]\oplus R_1[R_iR_j]}) \\
&\quad = \exp \mathrm{j}(\varphi_{(R_1\oplus R_2)[R_iR_j]} - \varphi_{(R_2\oplus R_1)[R_iR_j]}).
\end{aligned} \tag{30}
$$

Recalling (18) and (19) and noting, to be clearer $[R_i; R_j] = \exp \mathrm{j}\phi$ we obtain

$$(R_1 \oplus R_2)[R_iR_j] = \rho \exp \mathrm{j}(\varphi_{R_1\oplus R_2} + \phi) \tag{31}$$

$$(R_2 \oplus R_1)[R_iR_j] = \rho \exp \mathrm{j}(\varphi_{R_2\oplus R_1} + \phi) \tag{32}$$

so that the phase term in (30) becomes

$$\varphi_{(R_1\oplus R_2)[R_iR_j]} - \varphi_{(R_2\oplus R_1)[R_iR_j]} = \varphi_{R_1\oplus R_2} - \varphi_{R_2\oplus R_1}. \tag{33}$$

Inserting then (33) into (30) establishes the validity of W8. Note that this result is a special case of [4, equation 3.1V].

The loop property W10 can easily be shown. Using W3, we obtain

$$(R_1 \oplus R_2) \oplus R_2 = (R_2 \oplus (R_1 \oplus R_2))[R_1 \oplus R_2; R_2]. \tag{34}$$

Using successively W3 in the first line, W6 and W9 in the second line and W2*a* in the last line, the expression for $(R_1 \oplus R_2) \oplus R_3$ can also be written

$$
\begin{aligned}
(R_1 \oplus R_2) \oplus R_2 &= ((R_2 \oplus R_1)[R_1; R_2]) \oplus R_2 \\
&= ((R_2 \oplus R_1) \oplus R_2[R_2; R_1])[R_1; R_2] \\
&= (R_2 \oplus (R_1 \oplus R_2))[R_1; R_2].
\end{aligned} \tag{35}
$$

Comparison of the right-hand side of (34) and (35) establishes the validity of W10.

4. Calculation of partial reflection coefficients from a knowledge of the others

The group-like properties (W1)–(W5) show that the set of all complex amplitudes of the electromagnetic field in stratified planar structures forms a 'weakly associative–commutative groupoid' with the composition law $\oplus$. The knowledge of this gyrogroup permits deduction of any partial reflection coefficient (R_i or $\mathscr{R}_{(i,j,\ldots,n)}$) from a knowledge of others.

Performing such calculations needs to solve the basic equation

$$\mathscr{R}_{(1,2)} = R_1 \oplus R_2 \tag{36}$$

for R_1 or R_2.

(*a*) *Solving (36) for R_2.* To solve (36) for R_2 can easily be done. Composing (36) on the left with $(-R_1)$, and using W2*a* with W5 and W7 gives

$$\begin{aligned}(-R_1)\oplus\mathscr{R}_{(1,2)} &= (-R_1)\oplus(R_1\oplus R_2)\\ &= (-R_1\oplus R_1)\oplus R_2[-R_1;R_1]\\ &= R_2\end{aligned} \tag{37}$$

so that

$$\mathscr{R}_{(1,2)} = R_1\oplus R_2 \;\Rightarrow\; R_2 = (-R_1)\oplus\mathscr{R}_{(1,2)}. \tag{38}$$

(*b*) *Solving (36) for R_1.* To solve (36) for R_1 can be done in the same way. Multiplying (36) by $[R_2; R_1]$ and using W3 gives

$$\mathscr{R}_{(1,2)}[R_2;R_1] = R_2\oplus R_1. \tag{39}$$

The use of (37) then leads to

$$R_1 = (-R_2)\oplus(\mathscr{R}_{(1,2)}[R_2;R_1]). \tag{40}$$

5. Conclusion and discussion

The study of reflection of light by stratified planar structures leads to *a priori* unsuspected results.

The first result is that the reflection coefficient of any number of interfaces can be obtained directly by using a complex generalization of Einstein's addition law for parallel velocities. As explained in [1] this provides a useful mathematical tool in optics and in quantum theory (multiple quantum wells). Let us underline that the composition law of amplitudes may avoid some of the possible divergences which appear when adding probability amplitudes. Because of this, it may appear as a more natural 'addition law' of probability amplitudes.

The second result is that the set of all the complex amplitudes in stratified planar media form a weakly associative-distributive groupoid (a gyrogroup) with the composition law $\oplus$. This result is all the more unsuspected, because such an abstract mathematical structure, although discovered in 1965 by Karzel [7], only found its first concrete application three years ago when Ungar [2-6] showed that the set of all relativistically admissible velocities form a gyrogroup. Our present result constitutes another concrete example of this mathematical structure. The appearance of the same abstract structure in two different fields of physics becomes clearer when we note that reflection coefficients cannot exceed unity, exactly as velocities cannot exceed the speed of light.

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